The concept of energy is one of the most important topics in science and engineering. In everyday life, we think of energy in terms of fuel for transportation and heating, electricity for lights and appliances, and foods for consumption. However, these ideas do not really define energy. They merely tell us that fuels are needed to do a job and that those fuels provide us with something we call *energy*.

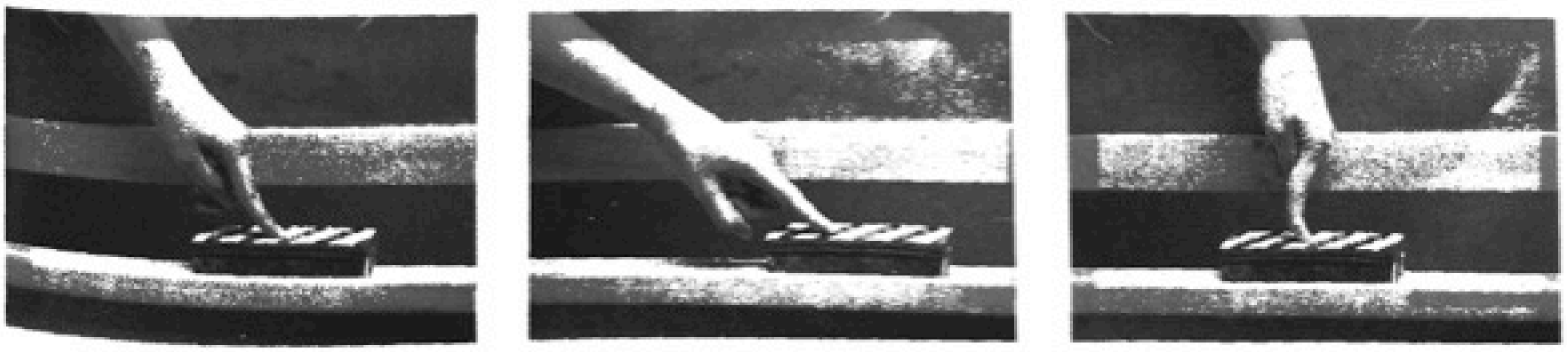
In this chapter, we first introduce the concept of work. Work is done by a force acting on an object when the point of application of that force moves through some distance and the force has a component along the line of motion. Next, we define kinetic energy, which is energy an object possesses because of its motion. In general, we can think of *energy* as the capacity that an object has for performing work. We shall see that the concepts of work and kinetic energy can be applied to the dynamics of a mechanical system without resorting to Newton's laws. In a complex situation, in fact, the "energy approach" can often allow a much simpler analysis than the direct application of Newton's second law. However, it is important to note that the work–energy concepts are based on Newton's laws and therefore allow us to make predictions that are always in agreement with these laws.

This alternative method of describing motion is especially useful when the force acting on a particle varies with the position of the particle. In this case, the acceleration is not constant, and we cannot apply the kinematic equations developed in Chapter 2. Often, a particle in nature is subject to a force that varies with the position of the particle. Such forces include the gravitational force and the force exerted on an object attached to a spring. Although we could analyze situations like these by applying numerical methods such as those discussed in Section 6.5, utilizing the ideas of work and energy is often much simpler. We describe techniques for treating complicated systems with the help of an extremely important theorem called the *work–kinetic energy theorem*, which is the central topic of this chapter.

7.1 WORK DONE BY A CONSTANT FORCE

5.1 Almost all the terms we have used thus far—velocity, acceleration, force, and so on—convey nearly the same meaning in physics as they do in everyday life. Now, however, we encounter a term whose meaning in physics is distinctly different from its everyday meaning. That new term is *work*.

To understand what *work* means to the physicist, consider the situation illustrated in Figure 7.1. A force is applied to a chalkboard eraser, and the eraser slides along the tray. If we are interested in how effective the force is in moving the



(a) (b) (c)

Figure 7.1 An eraser being pushed along a chalkboard tray. *(Charles D. Winters)* eraser, we need to consider not only the magnitude of the force but also its direction. If we assume that the magnitude of the applied force is the same in all three photographs, it is clear that the push applied in Figure 7.1b does more to move the eraser than the push in Figure 7.1a. On the other hand, Figure 7.1c shows a situation in which the applied force does not move the eraser at all, regardless of how hard it is pushed. (Unless, of course, we apply a force so great that we break something.) So, in analyzing forces to determine the work they do, we must consider the vector nature of forces. We also need to know how far the eraser moves along the tray if we want to determine the work required to cause that motion. Moving the eraser 3 m requires more work than moving it 2 cm.

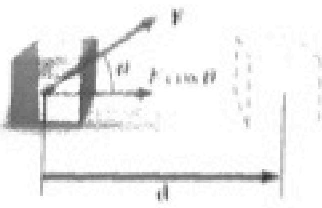


Figure 7.2 If an object undergoes a displacement **d** under the action of a constant force **F**, the work done by the force is $(F \cos \theta)d$.

Let us examine the situation in Figure 7.2, where an object undergoes a displacement **d** along a straight line while acted on by a constant force **F** that makes an angle θ with **d**.

Work done by a constant force

The **work** W done on an object by an agent exerting a constant force on the object is the product of the component of the force in the direction of the displacement and the magnitude of the displacement:

$$W = Fd \cos \theta \tag{7.1}$$

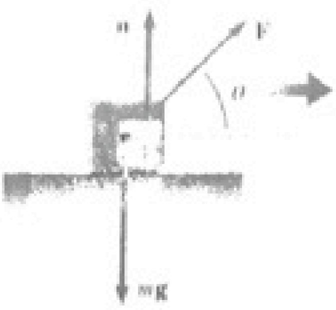


Figure 7.3 When an object is displaced on a frictionless, horizontal, surface, the normal force **n** and the force of gravity $m\mathbf{g}$ do no work on the object. In the situation shown here, **F** is the only force doing work on the object.

As an example of the distinction between this definition of work and our everyday understanding of the word, consider holding a heavy chair at arm's length for 3 min. At the end of this time interval, your tired arms may lead you to think that you have done a considerable amount of work on the chair. According to our definition, however, you have done no work on it whatsoever.[1] You exert a force to support the chair, but you do not move it. A force does no work on an object if the object does not move. This can be seen by noting that if $d = 0$, Equation 7.1 gives $W = 0$—the situation depicted in Figure 7.1c.

Also note from Equation 7.1 that the work done by a force on a moving object is zero when the force applied is perpendicular to the object's displacement. That is, if $\theta = 90°$, then $W = 0$ because $\cos 90° = 0$. For example, in Figure 7.3, the work done by the normal force on the object and the work done by the force of gravity on the object are both zero because both forces are perpendicular to the displacement and have zero components in the direction of **d**.

The sign of the work also depends on the direction of **F** relative to **d**. The work done by the applied force is positive when the vector associated with the component $F \cos \theta$ is in the same direction as the displacement. For example, when an object is lifted, the work done by the applied force is positive because the direction of that force is upward, that is, in the same direction as the displacement. When the vector associated with the component $F \cos \theta$ is in the direction opposite the displacement, W is negative. For example, as an object is lifted, the work done by the gravitational force on the object is negative. The factor $\cos \theta$ in the definition of W (Eq. 7.1) automatically takes care of the sign. It is important to note that **work is an energy transfer;** if energy is transferred *to* the system (object), W is positive; if energy is transferred *from* the system, W is negative.
5.3

[1] Actually, you do work while holding the chair at arm's length because your muscles are continuously contracting and relaxing; this means that they are exerting internal forces on your arm. Thus, work is being done by your body—but internally on itself rather than on the chair.

If an applied force **F** acts along the direction of the displacement, then $\theta = 0$ and $\cos 0 = 1$. In this case, Equation 7.1 gives

$$W = Fd$$

Work is a scalar quantity, and its units are force multiplied by length. Therefore, the SI unit of work is the **newton·meter** (N·m). This combination of units is used so frequently that it has been given a name of its own: the **joule** (J).

Quick Quiz 7.1

Can the component of a force that gives an object a centripetal acceleration do any work on the object? (One such force is that exerted by the Sun on the Earth that holds the Earth in a circular orbit around the Sun.)

In general, a particle may be moving with either a constant or a varying velocity under the influence of several forces. In these cases, because work is a scalar quantity, the total work done as the particle undergoes some displacement is the algebraic sum of the amounts of work done by all the forces.

EXAMPLE 7.1 Mr. Clean

A man cleaning a floor pulls a vacuum cleaner with a force of magnitude $F = 50.0$ N at an angle of 30.0° with the horizontal (Fig. 7.4a). Calculate the work done by the force on the vacuum cleaner as the vacuum cleaner is displaced 3.00 m to the right.

Solution Because they aid us in clarifying which forces are acting on the object being considered, drawings like Figure 7.4b are helpful when we are gathering information and organizing a solution. For our analysis, we use the definition of work (Eq. 7.1):

$$W = (F\cos\theta)d$$
$$= (50.0\ \text{N})(\cos 30.0°)(3.00\ \text{m}) = 130\ \text{N·m}$$
$$= 130\ \text{J}$$

One thing we should learn from this problem is that the normal force **n**, the force of gravity $\mathbf{F}_g = m\mathbf{g}$, and the upward component of the applied force (50.0 N) (sin 30.0°) do no work on the vacuum cleaner because these forces are perpendicular to its displacement.

Exercise Find the work done by the man on the vacuum cleaner if he pulls it 3.0 m with a horizontal force of 32 N.

Answer 96 J.

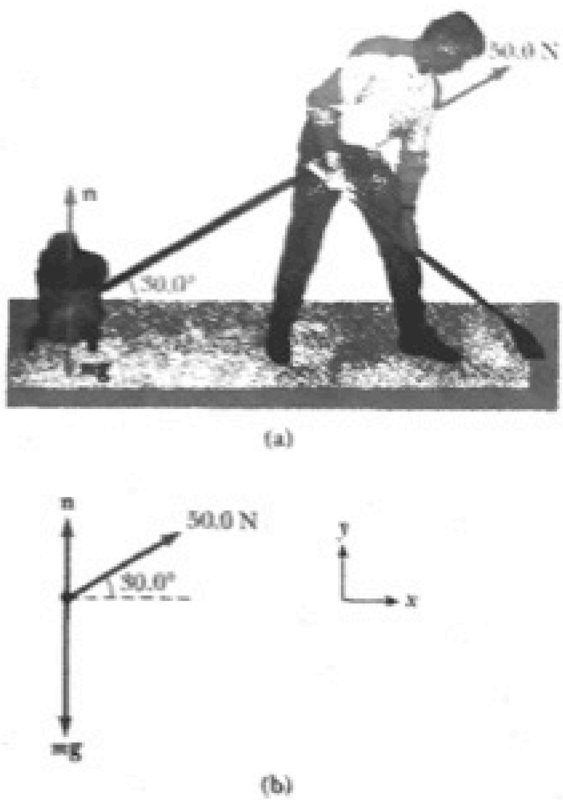
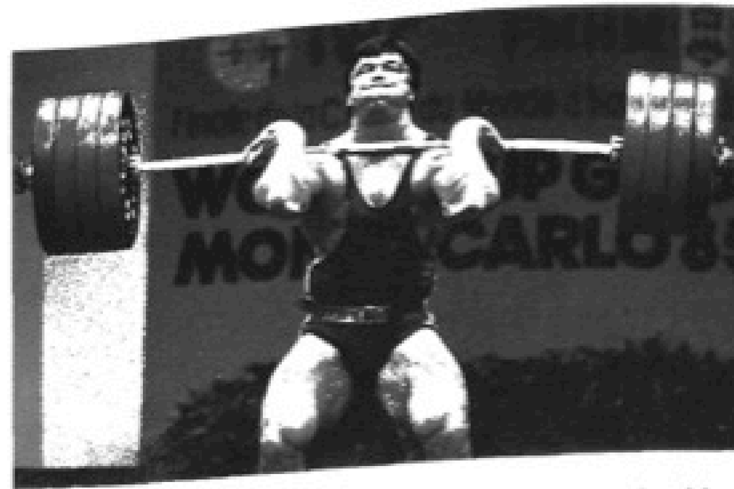


Figure 7.4 (a) A vacuum cleaner being pulled at an angle of 30.0° with the horizontal. (b) Free-body diagram of the forces acting on the vacuum cleaner.



The weightlifter does no work on the weights as he holds them on his shoulders. (If he could rest the bar on his shoulders and lock his knees, he would be able to support the weights for quite some time.) Did he do any work when he raised the weights to this height? *(Gerard Vandystadt/Photo Researchers, Inc.)*

Figure 7.5 A person lifts a box of mass m a vertical distance h and then walks horizontally a distance d.

Quick Quiz 7.2

A person lifts a heavy box of mass m a vertical distance h and then walks horizontally a distance d while holding the box, as shown in Figure 7.5. Determine (a) the work he does on the box and (b) the work done on the box by the force of gravity.

7.2 THE SCALAR PRODUCT OF TWO VECTORS

2.6 Because of the way the force and displacement vectors are combined in Equation 7.1, it is helpful to use a convenient mathematical tool called the **scalar product.** This tool allows us to indicate how **F** and **d** interact in a way that depends on how close to parallel they happen to be. We write this scalar product **F·d.** (Because of the dot symbol, the scalar product is often called the **dot product.**) Thus, we can express Equation 7.1 as a scalar product:

Work expressed as a dot product

$$W = \mathbf{F}\cdot\mathbf{d} = Fd\cos\theta \tag{7.2}$$

In other words, **F·d** (read "F dot d") is a shorthand notation for $Fd\cos\theta$.

Scalar product of any two vectors **A** and **B**

In general, the scalar product of any two vectors **A** and **B** is a scalar quantity equal to the product of the magnitudes of the two vectors and the cosine of the angle θ between them:

$$\mathbf{A}\cdot\mathbf{B} \equiv AB\cos\theta \tag{7.3}$$

This relationship is shown in Figure 7.6. Note that **A** and **B** need not have the same units.

In Figure 7.6, $B \cos\theta$ is the projection of **B** onto **A**. Therefore, Equation 7.3 says that $\mathbf{A}\cdot\mathbf{B}$ is the product of the magnitude of **A** and the projection of **B** onto **A**.[2]

From the right-hand side of Equation 7.3 we also see that the scalar product is **commutative.**[3] That is,

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$$

The order of the dot product can be reversed

Finally, the scalar product obeys the **distributive law of multiplication,** so that

$$\mathbf{A}\cdot(\mathbf{B} + \mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$$

The dot product is simple to evaluate from Equation 7.3 when **A** is either perpendicular or parallel to **B**. If **A** is perpendicular to **B** ($\theta = 90°$), then $\mathbf{A}\cdot\mathbf{B} = 0$. (The equality $\mathbf{A}\cdot\mathbf{B} = 0$ also holds in the more trivial case when either **A** or **B** is zero.) If vector **A** is parallel to vector **B** and the two point in the same direction ($\theta = 0$), then $\mathbf{A}\cdot\mathbf{B} = AB$. If vector **A** is parallel to vector **B** but the two point in opposite directions ($\theta = 180°$), then $\mathbf{A}\cdot\mathbf{B} = -AB$. The scalar product is negative when $90° < \theta < 180°$.

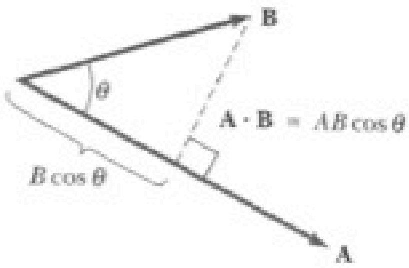


Figure 7.6 The scalar product $\mathbf{A}\cdot\mathbf{B}$ equals the magnitude of **A** multiplied by $B\cos\theta$, which is the projection of **B** onto **A**.

The unit vectors **i**, **j**, and **k**, which were defined in Chapter 3, lie in the positive *x*, *y*, and *z* directions, respectively, of a right-handed coordinate system. Therefore, it follows from the definition of $\mathbf{A}\cdot\mathbf{B}$ that the scalar products of these unit vectors are

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1 \tag{7.4}$$

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{i}\cdot\mathbf{k} = \mathbf{j}\cdot\mathbf{k} = 0 \tag{7.5}$$

Dot products of unit vectors

Equations 3.18 and 3.19 state that two vectors **A** and **B** can be expressed in component vector form as

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

$$\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$$

Using the information given in Equations 7.4 and 7.5 shows that the scalar product of **A** and **B** reduces to

$$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z \tag{7.6}$$

(Details of the derivation are left for you in Problem 7.10.) In the special case in which $\mathbf{A} = \mathbf{B}$, we see that

$$\mathbf{A}\cdot\mathbf{A} = A_x^2 + A_y^2 + A_z^2 = A^2$$

Quick Quiz 7.3

If the dot product of two vectors is positive, must the vectors have positive rectangular components?

[2] This is equivalent to stating that $\mathbf{A}\cdot\mathbf{B}$ equals the product of the magnitude of **B** and the projection of **A** onto **B**.

[3] This may seem obvious, but in Chapter 11 you will see another way of combining vectors that proves useful in physics and is not commutative.

EXAMPLE 7.2 The Scalar Product

The vectors **A** and **B** are given by $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{B} = -\mathbf{i} + 2\mathbf{j}$. (a) Determine the scalar product $\mathbf{A}\cdot\mathbf{B}$.

Solution

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= (2\mathbf{i} + 3\mathbf{j})\cdot(-\mathbf{i} + 2\mathbf{j})\\ &= -2\mathbf{i}\cdot\mathbf{i} + 2\mathbf{i}\cdot 2\mathbf{j} - 3\mathbf{j}\cdot\mathbf{i} + 3\mathbf{j}\cdot 2\mathbf{j}\\ &= -2(1) + 4(0) - 3(0) + 6(1)\\ &= -2 + 6 = 4\end{aligned}$$

where we have used the facts that $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = 1$ and $\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{i} = 0$. The same result is obtained when we use Equation 7.6 directly, where $A_x = 2$, $A_y = 3$, $B_x = -1$, and $B_y = 2$.

(b) Find the angle θ between **A** and **B**.

Solution The magnitudes of **A** and **B** are

$$A = \sqrt{A_x^2 + A_y^2} = \sqrt{(2)^2 + (3)^2} = \sqrt{13}$$

$$B = \sqrt{B_x^2 + B_y^2} = \sqrt{(-1)^2 + (2)^2} = \sqrt{5}$$

Using Equation 7.3 and the result from part (a) we find that

$$\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{AB} = \frac{4}{\sqrt{13}\sqrt{5}} = \frac{4}{\sqrt{65}}$$

$$\theta = \cos^{-1}\frac{4}{8.06} = 60.2^\circ$$

EXAMPLE 7.3 Work Done by a Constant Force

A particle moving in the xy plane undergoes a displacement $\mathbf{d} = (2.0\mathbf{i} + 3.0\mathbf{j})$ m as a constant force $\mathbf{F} = (5.0\mathbf{i} + 2.0\mathbf{j})$ N acts on the particle. (a) Calculate the magnitude of the displacement and that of the force.

Solution

$$d = \sqrt{x^2 + y^2} = \sqrt{(2.0)^2 + (3.0)^2} = 3.6\text{ m}$$

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{(5.0)^2 + (2.0)^2} = 5.4\text{ N}$$

(b) Calculate the work done by **F**.

Solution Substituting the expressions for **F** and **d** into Equations 7.4 and 7.5, we obtain

$$\begin{aligned}W = \mathbf{F}\cdot\mathbf{d} &= (5.0\mathbf{i} + 2.0\mathbf{j})\cdot(2.0\mathbf{i} + 3.0\mathbf{j})\text{ N}\cdot\text{m}\\ &= 5.0\mathbf{i}\cdot 2.0\mathbf{i} + 5.0\mathbf{i}\cdot 3.0\mathbf{j} + 2.0\mathbf{j}\cdot 2.0\mathbf{i} + 2.0\mathbf{j}\cdot 3.0\mathbf{j}\\ &= 10 + 0 + 0 + 6 = 16\text{ N}\cdot\text{m} = 16\text{ J}\end{aligned}$$

Exercise Calculate the angle between **F** and **d**.

Answer 35°.

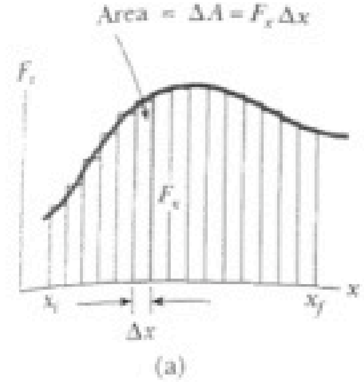
7.3 WORK DONE BY A VARYING FORCE

5.2 Consider a particle being displaced along the x axis under the action of a varying force. The particle is displaced in the direction of increasing x from $x = x_i$ to $x = x_f$. In such a situation, we cannot use $W = (F\cos\theta)d$ to calculate the work done by the force because this relationship applies only when **F** is constant in magnitude and direction. However, if we imagine that the particle undergoes a very small displacement Δx, shown in Figure 7.7a, then the x component of the force F_x is approximately constant over this interval; for this small displacement, we can express the work done by the force as

$$\Delta W = F_x\,\Delta x$$

This is just the area of the shaded rectangle in Figure 7.7a. If we imagine that the F_x versus x curve is divided into a large number of such intervals, then the total work done for the displacement from x_i to x_f is approximately equal to the sum of a large number of such terms:

$$W \approx \sum_{x_i}^{x_f} F_x\,\Delta x$$



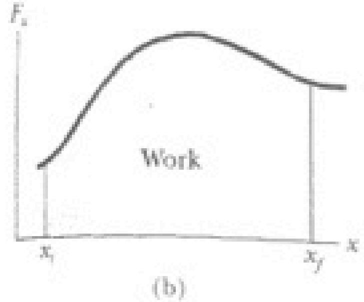


Figure 7.7 (a) The work done by the force component F_x for the small displacement Δx is $F_x \Delta x$, which equals the area of the shaded rectangle. The total work done for the displacement from x_i to x_f is approximately equal to the sum of the areas of all the rectangles. (b) The work done by the component F_x of the varying force as the particle moves from x_i to x_f is exactly equal to the area under this curve.

If the displacements are allowed to approach zero, then the number of terms in the sum increases without limit but the value of the sum approaches a definite value equal to the area bounded by the F_x curve and the x axis:

$$\lim_{\Delta x \to 0} \sum_{x_i}^{x_f} F_x \Delta x = \int_{x_i}^{x_f} F_x \, dx$$

This definite integral is numerically equal to the area under the F_x-versus-x curve between x_i and x_f. Therefore, we can express the work done by F_x as the particle moves from x_i to x_f as

$$W = \int_{x_i}^{x_f} F_x \, dx \qquad (7.7)$$

Work done by a varying force

This equation reduces to Equation 7.1 when the component $F_x = F \cos \theta$ is constant.

If more than one force acts on a particle, the total work done is just the work done by the resultant force. If we express the resultant force in the x direction as ΣF_x, then the total work, or *net work*, done as the particle moves from x_i to x_f is

$$\sum W = W_{\text{net}} = \int_{x_i}^{x_f} \left(\sum F_x \right) dx \qquad (7.8)$$

EXAMPLE 7.4 **Calculating Total Work Done from a Graph**

A force acting on a particle varies with x, as shown in Figure 7.8. Calculate the work done by the force as the particle moves from $x = 0$ to $x = 6.0$ m.

Solution The work done by the force is equal to the area under the curve from $x_A = 0$ to $x_C = 6.0$ m. This area is equal to the area of the rectangular section from Ⓐ to Ⓑ plus

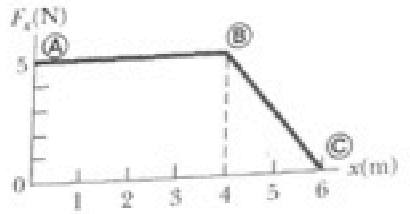


Figure 7.8 The force acting on a particle is constant for the first 4.0 m of motion and then decreases linearly with x from $x_B = 4.0$ m to $x_C = 6.0$ m. The net work done by this force is the area under the curve.

the area of the triangular section from Ⓑ to Ⓒ. The area of the rectangle is (4.0)(5.0) N·m = 20 J, and the area of the triangle is $\frac{1}{2}$(2.0)(5.0) N·m = 5.0 J. Therefore, the total work done is 25 J.

EXAMPLE 7.5 Work Done by the Sun on a Probe

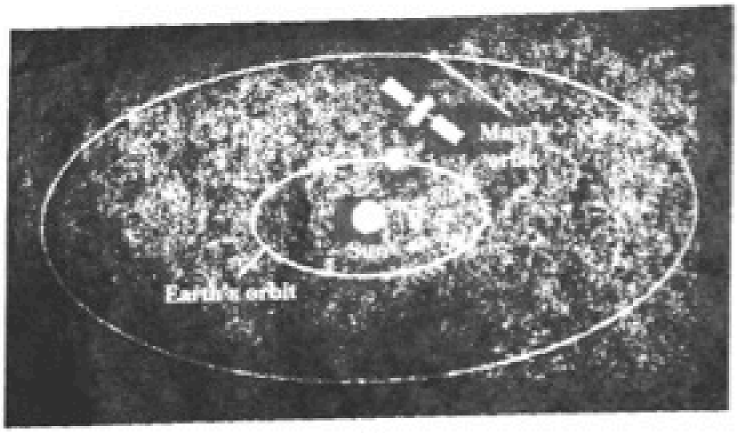
The interplanetary probe shown in Figure 7.9a is attracted to the Sun by a force of magnitude

$$F = -1.3 \times 10^{22}/x^2$$

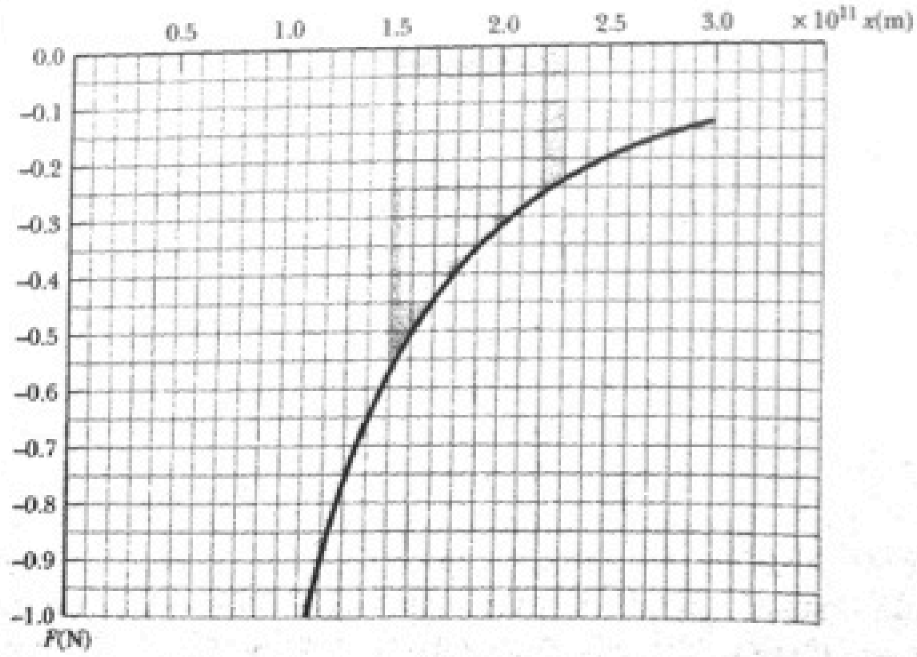
where x is the distance measured outward from the Sun to the probe. Graphically and analytically determine how much work is done by the Sun on the probe as the probe–Sun separation changes from 1.5×10^{11} m to 2.3×10^{11} m.

Graphical Solution The minus sign in the formula for the force indicates that the probe is attracted to the Sun. Because the probe is moving away from the Sun, we expect to calculate a negative value for the work done on it.

A spreadsheet or other numerical means can be used to generate a graph like that in Figure 7.9b. Each small square of the grid corresponds to an area $(0.05\ \text{N})(0.1 \times 10^{11}\ \text{m}) = 5 \times 10^8$ N·m. The work done is equal to the shaded area in Figure 7.9b. Because there are approximately 60 squares shaded, the total area (which is negative because it is below the x axis) is about -3×10^{10} N·m. This is the work done by the Sun on the probe.



(a)



(b)

Figure 7.9 (a) An interplanetary probe moves from a position near the Earth's orbit radially outward from the Sun, ending up near the orbit of Mars. (b) Attractive force versus distance for the interplanetary probe.

Analytical Solution We can use Equation 7.7 to calculate a more precise value for the work done on the probe by the Sun. To solve this integral, we use the first formula of Table B.5 in Appendix B with $n = -2$:

$$W = \int_{1.5\times10^{11}}^{2.3\times10^{11}} \left(\frac{-1.3 \times 10^{22}}{x^2}\right) dx$$

$$= (-1.3 \times 10^{22}) \int_{1.5\times10^{11}}^{2.3\times10^{11}} x^{-2}\, dx$$

$$= (-1.3 \times 10^{22})(-x^{-1}) \Big|_{1.5\times10^{11}}^{2.3\times10^{11}}$$

$$= (-1.3 \times 10^{22})\left(\frac{-1}{2.3 \times 10^{11}} - \frac{-1}{1.5 \times 10^{11}}\right)$$

$$= -3.0 \times 10^{10}\ \text{J}$$

Exercise Does it matter whether the path of the probe is not directed along a radial line away from the Sun?

Answer No; the value of W depends only on the initial and final positions, not on the path taken between these points.

Work Done by a Spring

A common physical system for which the force varies with position is shown in Figure 7.10. A block on a horizontal, frictionless surface is connected to a spring. If the spring is either stretched or compressed a small distance from its unstretched (equilibrium) configuration, it exerts on the block a force of magnitude

$$F_s = -kx \tag{7.9}$$

Spring force

where x is the displacement of the block from its unstretched ($x = 0$) position and k is a positive constant called the **force constant** of the spring. In other words, the force required to stretch or compress a spring is proportional to the amount of stretch or compression x. This force law for springs, known as **Hooke's law,** is valid only in the limiting case of small displacements. The value of k is a measure of the *stiffness* of the spring. Stiff springs have large k values, and soft springs have small k values.

Quick Quiz 7.4

What are the units for k, the force constant in Hooke's law?

The negative sign in Equation 7.9 signifies that the force exerted by the spring is always directed *opposite* the displacement. When $x > 0$ as in Figure 7.10a, the spring force is directed to the left, in the negative x direction. When $x < 0$ as in Figure 7.10c, the spring force is directed to the right, in the positive x direction. When $x = 0$ as in Figure 7.10b, the spring is unstretched and $F_s = 0$. Because the spring force always acts toward the equilibrium position ($x = 0$), it sometimes is called a *restoring force.* If the spring is compressed until the block is at the point $-x_{max}$ and is then released, the block moves from $-x_{max}$ through zero to $+x_{max}$. If the spring is instead stretched until the block is at the point x_{max} and is then released, the block moves from $+x_{max}$ through zero to $-x_{max}$. It then reverses direction, returns to $+x_{max}$, and continues oscillating back and forth.

Suppose the block has been pushed to the left a distance x_{max} from equilibrium and is then released. Let us calculate the work W_s done by the spring force as the block moves from $x_i = -x_{max}$ to $x_f = 0$. Applying Equation 7.7 and assuming the block may be treated as a particle, we obtain

$$W_s = \int_{x_i}^{x_f} F_s\, dx = \int_{-x_{max}}^{0} (-kx)\, dx = \tfrac{1}{2}kx_{max}^2 \tag{7.10}$$

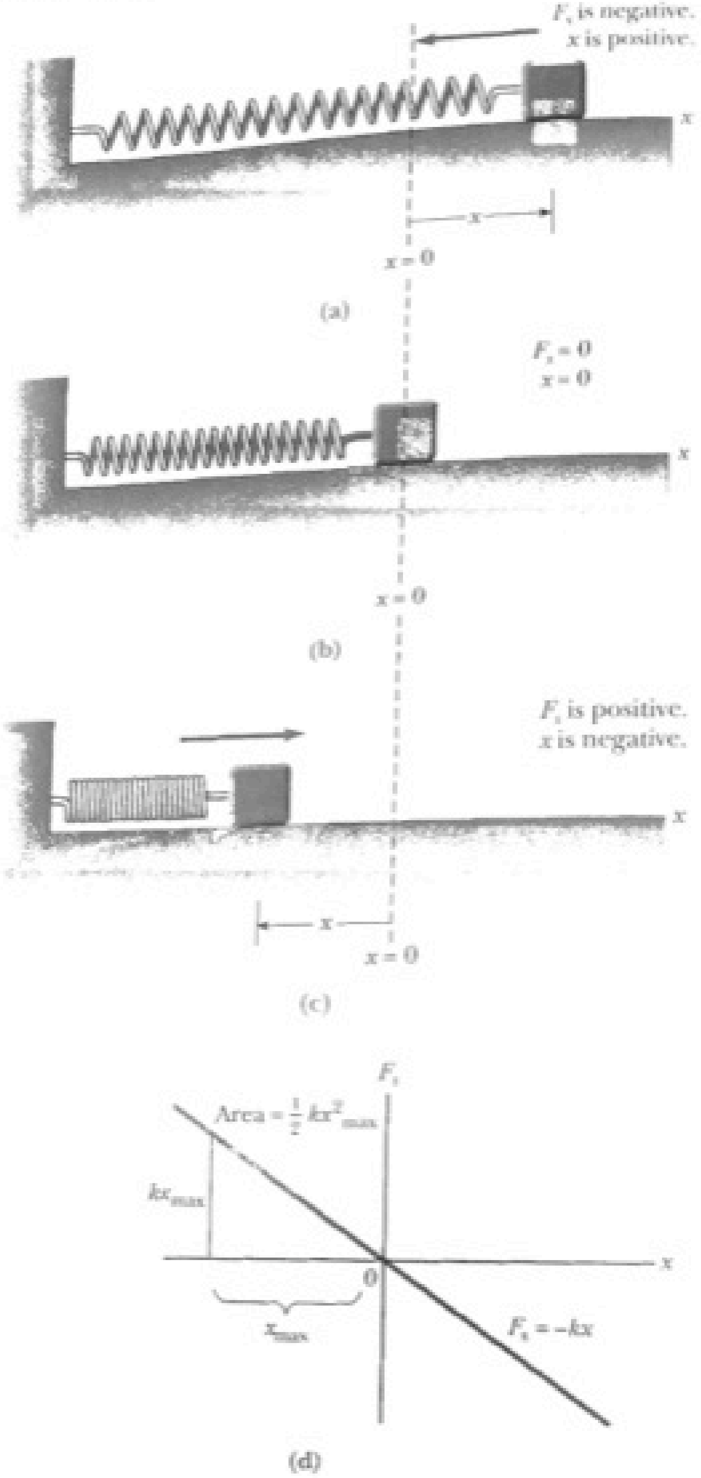


Figure 7.10 The force exerted by a spring on a block varies with the block's displacement x from the equilibrium position $x = 0$. (a) When x is positive (stretched spring), the spring force is directed to the left. (b) When x is zero (natural length of the spring), the spring force is zero. (c) When x is negative (compressed spring), the spring force is directed to the right. (d) Graph of F_s versus x for the block–spring system. The work done by the spring force as the block moves from $-x_{max}$ to 0 is the area of the shaded triangle, $\frac{1}{2}kx^2_{max}$.

where we have used the indefinite integral $\int x^n dx = x^{n+1}/(n + 1)$ with $n = 1$. The work done by the spring force is positive because the force is in the same direction as the displacement (both are to the right). When we consider the work done by the spring force as the block moves from $x_i = 0$ to $x_f = x_{max}$, we find that

$W_s = -\frac{1}{2}kx_{max}^2$ because for this part of the motion the displacement is to the right and the spring force is to the left. Therefore, the *net* work done by the spring force as the block moves from $x_i = -x_{max}$ to $x_f = x_{max}$ is zero.

Figure 7.10d is a plot of F_s versus x. The work calculated in Equation 7.10 is the area of the shaded triangle, corresponding to the displacement from $-x_{max}$ to 0. Because the triangle has base x_{max} and height kx_{max}, its area is $\frac{1}{2}kx_{max}^2$, the work done by the spring as given by Equation 7.10.

If the block undergoes an arbitrary displacement from $x = x_i$ to $x = x_f$, the work done by the spring force is

$$W_s = \int_{x_i}^{x_f} (-kx)\,dx = \tfrac{1}{2}kx_i^2 - \tfrac{1}{2}kx_f^2 \qquad \textbf{(7.11)}$$

Work done by a spring

For example, if the spring has a force constant of 80 N/m and is compressed 3.0 cm from equilibrium, the work done by the spring force as the block moves from $x_i = -3.0$ cm to its unstretched position $x_f = 0$ is 3.6×10^{-2} J. From Equation 7.11 we also see that the work done by the spring force is zero for any motion that ends where it began ($x_i = x_f$). We shall make use of this important result in Chapter 8 in which we describe the motion of this system in greater detail.

Equations 7.10 and 7.11 describe the work done by the spring on the block. Now let us consider the work done on the spring by an *external agent* that stretches the spring very slowly from $x_i = 0$ to $x_f = x_{max}$, as in Figure 7.11. We can calculate this work by noting that at any value of the displacement, the *applied force* $\mathbf{F}_{app}$ is equal to and opposite the spring force $\mathbf{F}_s$, so that $F_{app} = -(-kx) = kx$. Therefore, the work done by this applied force (the external agent) is

$$W_{F_{app}} = \int_0^{x_{max}} F_{app}\,dx = \int_0^{x_{max}} kx\,dx = \tfrac{1}{2}kx_{max}^2$$

This work is equal to the negative of the work done by the spring force for this displacement.

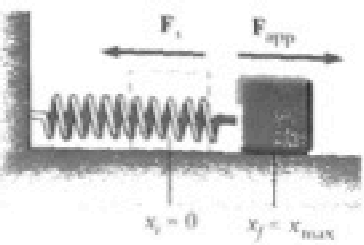


Figure 7.11 A block being pulled from $x_i = 0$ to $x_f = x_{max}$ on a frictionless surface by a force $\mathbf{F}_{app}$. If the process is carried out very slowly, the applied force is equal to and opposite the spring force at all times.

EXAMPLE 7.6 Measuring k for a Spring

A common technique used to measure the force constant of a spring is described in Figure 7.12. The spring is hung vertically, and an object of mass m is attached to its lower end. Under the action of the "load" mg, the spring stretches a distance d from its equilibrium position. Because the spring force is upward (opposite the displacement), it must balance the downward force of gravity mg when the system is at rest. In this case, we can apply Hooke's law to give $|\mathbf{F}_s| = kd = mg$, or

$$k = \frac{mg}{d}$$

For example, if a spring is stretched 2.0 cm by a suspended object having a mass of 0.55 kg, then the force constant is

$$k = \frac{mg}{d} = \frac{(0.55\text{ kg})(9.80\text{ m/s}^2)}{2.0 \times 10^{-2}\text{ m}} = 2.7 \times 10^2\text{ N/m}$$

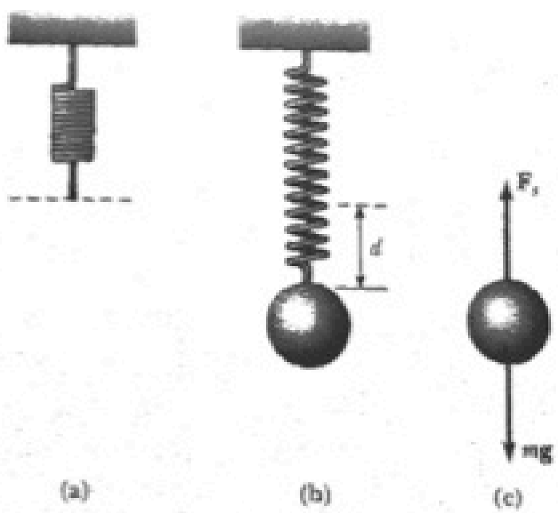


Figure 7.12 Determining the force constant k of a spring. The elongation d is caused by the attached object, which has a weight mg. Because the spring force balances the force of gravity, it follows that $k = mg/d$.